# OSCILLATIONS OF FIRST ORDER NEUTRAL IMPULSIVE DELAY DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

by

M.K. Grammatikopoulos[(1)], M.B. Dimitrova[(2)] and V.I. Donev[(2)]

Abstract. This paper is dealing with the oscillatory properties of first order delay neutral impulsive differential equations and corresponding to them inequalities with constant coefficients. The established sufficient conditions ensure the oscillation of every solution of this type of equations.

**Key words and phrases:** oscillation of solutions, neutral impulsive delay differential equations and inequalities, constant coefficients.

**AMS (MOS) Subject Classifications:** 34K11, 34K40, 34A37.

## 1. Introduction

Impulsive differential equations with deviating arguments (IDEDA) are adequate mathematical models for the simulation of processes that depend on their history and are subject to short-time disturbances. Such processes occur in the theory of optimal control, theoretical physics, population dynamics, biotechnology, industrial robotics, etc. In contrast to the theory of ordinary impulsive differential equations (see, [1] - [3] and [20] ) and differential equations with deviating arguments (see, [11], [13], [14]and [18]), the theory of IDEDA admits some theoretical and practical difficulties. We note here that [12] is the first work where IDEDA were considered. For more results, concerning IDEDA, we choose to refer to [4]-[6],[8],[23] and [24]. Much less we know about the neutral impulsive differential equations, i.e. equations in which the highest-order derivative of the unknown function appears in the equation with the argument t (the present state of the system), as well as with one or more retarded and/or advanced arguments (the past and/or the future state of the system). Note that equations of this type appear in networks, containing lossless transmission lines. Such networks arise , for example, in high speed computers, where lossless transmission lines are used to interconnect switching circuits (see, [7] and [21]).

As it is known (see [11]), the appearance of the neutral term in a differential equation can cause or destroy the oscillation of its solutions. Moreover, the study of neutral differential equations in general , presents complications which are unfamiliar for non-neutral differential equations. As far as for a discussion on some more applications and some drastic differences in behavior of the solution of neutral differential equations see, for example, [15],[16] and [22].

## 2. Preliminaries

In this article we consider the first order delay neutral impulsive differential equation with constant coefficients of the form

$$\frac{d}{dt}[y(t) - cy(t-h)] + qy(t-\sigma) = 0, \ t \neq \tau_k \tag{$E_1$}$$

$$\Delta[y(\tau_k) - cy(\tau_k - h)] + p_k y(\tau_k - \sigma) = 0, \ k \in N$$

as well as the corresponding to it inequalities

$$\frac{d}{dt}[y(t) - cy(t-h)] + qy(t-\sigma) \leq 0, \ t \neq \tau_k \tag{$N_{1,\leq}$}$$

$$\Delta[y(\tau_k) - cy(\tau_k - h)] + p_k y(\tau_k - \sigma) \leq 0, \ k \in N$$

and

$$\frac{d}{dt}[y(t) - cy(t-h)] + qy(t-\sigma) \geq 0, \ t \neq \tau_k \tag{$N_{1,\geq}$}$$

$$\Delta[y(\tau_k) - cy(\tau_k - h)] + p_k y(\tau_k - \sigma) \geq 0, \ k \in N$$

where $c \in (0,1)$, $q$, $p_k \in [0, +\infty), k \in N$ and $h, \sigma \in (0, +\infty)$.

Moreover, we consider a special case of the equation $(E_1)$ and the corresponding to it inequalities which are of the form

$$y'(t) + qy(t-\sigma) = 0, \ t \neq \tau_k \tag{$E_2$}$$

$$\Delta y(\tau_k) + p_k y(\tau_k - \sigma) = 0, \ k \in N$$

$$y'(t) + qy(t-\sigma) \leq 0, \ t \neq \tau_k \tag{$N_{2,\leq}$}$$

$$\Delta y(\tau_k) + p_k y(\tau_k - \sigma) \leq 0, \ k \in N$$

and

$$y'(t) + qy(t-\sigma) \geq 0, \ t \neq \tau_k \tag{$N_{2,\geq}$}$$

$$\Delta y(\tau_k) + p_k y(\tau_k - \sigma) \geq 0, \ k \in N$$

respectively.

Here the deviations $h$ and/or $\sigma$ are positive constants and $\tau_k \in (0, +\infty), k \in N$ are fixed moments of impulsive effect (the jump points), which we characterize as *down-jumps* when $\Delta x(\tau_k) < 0$, $k \in N$ and as *up-jumps* when $\Delta x(\tau_k) > 0$, $k \in N$.

Denote by $PC(R,R)$ the set of all piecewise continuous on the intervals $(\tau_k,\tau_{k+1}]$, $k\in N$ functions $u\colon R\to R$ which at the jump points $\tau_k$, $k\in N$ are continuous from the left, i.e. $u(\tau_k-0)=\lim_{t\to\tau_k-0}u(t)=u(\tau_k)$, and may have discontinuities of first kind at the jump points $\tau_k$, $k\in N$.

Suppose that the fixed moments of impulsive effect (the jump points) $\tau_k$, $k\in N$ have the properties

$$t_0=\tau_0<\tau_1<\tau_2<\ldots<\tau_k<\ldots,\ \lim_{k\to+\infty}\tau_k=+\infty,\max\{\tau_{k+1}-\tau_k\}<+\infty,\ k\in N$$

Moreover, the following notions will be used throughout this paper.

A continuous real valued function $u$ defined on an interval of the form $[a,+\infty)$ eventually has some property if there is a number $b\geq a$ such that $u$ has this property on the interval $[b,+\infty)$.

Let $\rho=\max\{\sigma,h\}$. We will say that a function $y(t)$ is a solution of $Eq.(E_1)$, if there exists a number $T_0\in R$ such that $y\in PC([T_0-\rho,+\infty],R)$, the function $z(t)=y(t)-cy(t-h)$ is continuously differentiable for $t\geq T_0$, $t\neq\tau_k$, $k\in N$ and $y(t)$ satisfies $Eq.(E_1)$ for all $t\geq T_0$.

Furthermore, our results here pertain only to the nontrivial continuable solutions $y(t)$ of the equation $(E_1)$, i.e. $y(t)$ is defined on an interval of the form $[T_y,+\infty)$ for some $T_y\geq T_0$ and

$$\sup\{|y(t)|\colon\ t\geq T\}>0 \text{ for each } T\geq T_y.$$

Such a solution of $Eq.(E_1)$ is called *regular*. A regular solution $y(t)$ of $Eq.(E_1)$, is said to be *nonoscillatory,* if there exists a number $t_0\geq 0$ such that $y(t)$ is of constant sign for every $t\geq t_0$. Otherwise, it is called *oscillatory*. Also, note that a *nonoscillatory* solution is called *eventually positive* (*eventually negative*), if the constant sign that determines its *nonoscillation* is positive (negative). Equation $(E_1)$ is called *oscillatory,* if all its solutions are oscillatory. Otherwise, it is called *nonoscillatory.*

In what follows we will consider $Eq.(E_1)$, only in the cases, where it is a **neutral** ($h\neq 0$, $c\neq 0$) and an **impulsive** ($p_k\neq 0$ or $p_k=0$ with $\tau_{k+1}-\tau_k=h$, $k\in N$) **differential equation** with two different deviations ($\sigma\neq 0$, $h\neq 0$, $\sigma\neq h$) or with a single deviation ($\sigma=h\neq 0$). So, in what follows, without further mention, we will assume that

$$c\in(0,1),\ q,p_k\in[0,+\infty),k\in N \text{ and } h,\sigma\in(0,+\infty)$$

Finally, in this article, when we write a functional expression, we will mean that it holds for all sufficiently large values of the argument.

Our aim is to establish sufficient conditions under which the equation $(E_1)$ is oscillatory.

To this end, we need the following two lemmas.

The first lemma (see, [9],[10] and[13]) describes the asymptotic behavior of the functions $z(t)=y(t)-cy(t-h)$ and $y(t)$, where $y(t)$ is an eventually positive solution of $Eq.(E_1)$.

**Lemma 1** *Let $y(t)$ be an eventually positive solution of $Eq.(E_1)$. Then:*

*(a) $z(t) > 0$ for all large $t$ with $\lim_{t\to+\infty} z(t) = 0$ and $\lim_{\tau_k\to+\infty} |\Delta z(\tau_k)| = 0$;*

*(b) $\lim_{t\to+\infty} y(t) = 0$ and $\lim_{\tau_k\to+\infty} |\Delta y(\tau_k)| = 0$.*

Lemma 1, applied to the differentiable function $z(t) = y(t) - cy(t-h)$ and to twice differentiable function $w(t) = z(t) - cz(t-h)$, where $y(t)$ is an eventually positive solution of $Eq.(E_1)$, leads to the following proposition which is useful for our purposes.

**Lemma 2** *Let $y(t)$ be an eventually positive solution of $Eq.(E_1)$. Then the functions $z(t) = y(t) - cy(t-h)$ and $w(t) = z(t) - cz(t-h)$ are also solutions of $Eq.(E_1)$ with the properties:*

*(a) $z(t) > 0$, $z'(t) < 0$ eventually and*

$$\lim_{t\to+\infty} z(t) = 0, \lim_{\tau_k\to+\infty} |\Delta z(\tau_k)| = 0;$$

*(b) $w(t) > 0$, $w'(t) < 0$ and $w''(t) > 0$ eventually and*

$$\lim_{t\to+\infty} w(t) = 0, \lim_{\tau_k\to+\infty} |\Delta w(\tau_k)| = 0.$$

**Proof.** As the negative of a solution of $(E_1)$ is also a solution of the same equation, it suffices to prove the lemma for an eventually positive solution y(t) of $(E_1)$. Thus, assume, for the sake of contradiction, that $y(t)$ is an eventually positive solution of $(E_1)$. Then, since the equation $(E_1)$ is an autonomous one, it follows that $y(t-h)$ is also a solution of $(E_1)$. Therefore, $z(t)$ as a linear combination of solutions of $(E_1)$ is itself a solution of $(E_1)$. By similar arguments we easily conclude that $w(t)$ is also a solution of $(E_1)$. Now, using Lemma 1, it is easy to see that for all large $t$

$$z(t) > 0, \; z'(t) < 0$$

and that

$$\lim_{t\to+\infty} z(t) = 0, \lim_{\tau_k\to+\infty} |\Delta z(\tau_k)| = 0.$$

By the same manner we conclude that for all large $t$

$$w(t) > 0, \; w'(t) < 0 \; and \; w(t)'' = [z(t) - cz(t-h)]' = -qz'(t-\sigma) > 0, \; t \neq \tau_k$$

and that

$$\lim_{t\to+\infty} w(t) = 0, \lim_{\tau_k\to+\infty} |\Delta w(\tau_k)| = 0.$$

This completes the proof of the lemma.

## 3. Oscillation of all solutions of $(E_2)$

The results of this section will be used in the study of the oscillatory properties of $(E_1)$ and the corresponding to it inequalities $(N_{1,\leq})$ and $(N_{1,\geq})$ respectively.

Consider the first order ordinary impulsive delay differential equation $(E_2)$ and the corresponding to it inequalities $(N_{2,\leq})$ and $(N_{2,\geq})$, which are special cases of the equation $(E_1)$.

Note that, as it is well-known (see, for example, [20] and [18]), a necessary and sufficient condition for the oscillation of all solutions of the delay differential equation $(E_2)$, without impulsive effects, is that $q\sigma > \frac{1}{e}$. On the other hand, if the condition $q\sigma \leq \frac{1}{e}$ holds, then, according to a result in [17] (see also [18]), the delay differential equation $(E_2)$, without impulsive effects, is non-oscillatory. Our results below, demonstrate the influence of impulsive effects on the behavior of solutions of $(E_2)$. Indeed, Corollary 1 below shows the fact that the delay differential equation $(E_2)$, subject to impulsive effects, is oscillatory even in the case, where $q\sigma \leq \frac{1}{e}$.

**Theorem 1** *Assume that*

$$\liminf_{t\to+\infty}\Big(q\sigma+\sum_{t-\sigma\leq\tau_k\leq t} p_k\Big) \geq 1.$$

*Then:*

*(a) the equation $(E_2)$ is oscillatory;*

*(b) the inequality $(N_{2,\leq})$ has no eventually positive solutions;*

*(c) the inequality $(N_{2,\geq})$, has no eventually negative solutions.*

**Proof.** Since the proofs of (a),(b) and (c) can be carried out by similar arguments, it suffices to prove only the case (a). To this end, as in the proof of Lemma 2, we assume that y(t) is an eventually positive solution of $(E_2)$. Then there exists a $t_0 > 0$ such that $y(t) > 0$ for every $t > t_0$. Also, there is a $t_1 \geq t_0+\sigma$ such that $y(t-\sigma) > 0$, $y'(t) < 0$ and $\Delta y(\tau_k) = -p_k y(\tau_k-\sigma) < 0$, $k \in N$ for every $t \geq t_1$. That means that $y$ is decreasing function with down-jumps $(\Delta y(\tau_k) < 0)$, $k \in N$.

Integrating $(E_2)$ from $t-\sigma$ to $t$, we find

$$y(t) - y(t-\sigma) - \sum_{t-\sigma\leq\tau_k\leq t} \Delta y(\tau_k) + \int_{t-\sigma}^{t} qy(s-\sigma)ds = 0.$$

Remark that, because $y(t)$ is a positive decreasing function of $t$ , from last equality we derive

$$-y(t-\sigma) + \sum_{t-\sigma\leq\tau_k\leq t} p_k y(\tau_k-\sigma) + q\sigma y(t-\sigma) \leq 0. \tag{1}$$

as well as

$$y(\tau_k-\sigma) > y(t-\sigma) > 0, \quad \text{when } \tau_k-\sigma < t-\sigma.$$

Hence, (1) yields

$$y(t-\sigma)(-1+q\sigma+\sum_{t-\sigma\le\tau_k<t} p_k)<0$$

and finally we conclude that

$$q\sigma+\sum_{t-\sigma\le\tau_k\le t} p_k<1.$$

But the last inequality contradicts our assumptions and the conclusion of the theorem is evident.

As a consequence of the above theorem, we have the following important

**Corollary 1** *Let* $0\le q\sigma\le\frac{1}{e}$ *and assume that* $\liminf_{t\to+\infty}\sum_{t-\sigma\le\tau_k\le t} p_k\ge 1$*.*

*Then the conclusion of Theorem 1 holds.*

We conclude this section with the following

## 4. Oscillation of all solutions of Eq. $(E_1)$.

Having in mind the results of the previous section, we establish our main result which ensure the oscillation of all solutions of the equation $(E_1)$.

**Theorem 2** *Assume that* $\sigma>h$ *and that*

$$\liminf_{t\to+\infty}[q(\sigma-h)+\sum_{t-(\sigma-h)\le\tau_k\le t} p_k]\ge 1-c.$$

*Then:*

*(a) the equation (1) is oscillatory;*

*(b) the inequality (2) has now eventually positive solutions;*

*(c) the inequality (3) has no eventually negative solutions.*

**Proof.** As in the proof of Theorem 1, we prove only the case (a). To do that, as in the proof of Lemma 2, we assume, for the sake of contradiction, that Eq. $(E_1)$ has an eventually positive solution $y(t)$. Then there exists a $t_0>0$ such that $y(t)>0$ for every $t>t_0$. Also, there is a $t_1\ge t_0+\sigma$ such that $y(t-\sigma)>0$, $y'(t)<0$ and $\Delta[y(\tau_k)-cy(\tau_k-h)]=-p_k y(\tau_k-\sigma)<0$, $k\in N$ for every $t\ge t_1$. Now, by Lemma 2, it follows that for every $t\ge t_1$ the functions $z(t)=y(t)-cy(t-h)>0$ and $w(t)=z(t)-cz(t-h)>0$ are solutions to the equation $(E_1)$. That is, $w(t)$ satisfies the equation

$$\frac{d}{dt}[w(t)-cw(t-h)]+qw(t-\sigma)=0,\ t\ne\tau_k, \qquad (2)$$

$$\Delta[w(\tau_k) - cw(\tau_k - h)] + p_k w(\tau_k - \sigma) = 0,\ k \in N$$

Note that, by Lemma 2, $w(t)$ is an eventually positive strongly decreasing, while $w'(t)$ is an eventually negative strongly increasing function. Therefore, it is easy to see that

$$\begin{aligned} w'(t-h) - cw'(t-h) + qw(t-\sigma+h) &\leq w'(t) - cw'(t-h) + qw(t-\sigma) \\ &= \frac{d}{dt}[w(t) - cw(t-h)] + qw(t-\sigma) = 0 \end{aligned} \tag{3}$$

Moreover, since $z(t)$ is a decreasing function, we see that $z(\tau_k - \sigma) < z(\tau_k - \sigma - h)$ and so, using the definitions of the functions $z(t)$ and $w(t)$, it is easy to conclude that

$$\Delta w(\tau_k) = -p_k z(\tau_k - \sigma) > -p_k z(\tau_k - \sigma - h) = \Delta w(\tau_k - h),\ k \in N$$

So, in view of the above observation, from (2) it follows that for each $k \in N$

$$\begin{aligned} \Delta w(\tau_k - h) - c\Delta w(\tau_k - h) + p_k w(\tau_k - \sigma + h) &\leq \Delta w(\tau_k) - c\Delta w(\tau_k - h) + p_k w(\tau_k - \sigma) \\ &= \Delta[w(\tau_k) - cw(\tau_k - h)] + p_k w(\tau_k - \sigma) = 0 \end{aligned} \tag{4}$$

Now, by (3) and (4) , it follows that $w(t)$ is an eventually positive function for which

$$(1-c)w'(t-h) + qw(t-\sigma+h) \leq 0,\ t \neq \tau_k$$

$$(1-c)\Delta w(\tau_k - h) + p_k w(\tau_k - \sigma + h) \leq 0,\ k \in N$$

Hence, we conclude that $w(t)$ is an eventually positive solution to the inequality

$$w'(t) + \frac{q}{1-c} w(t-\sigma+h) \leq 0,\ t \neq \tau_k, \tag{5}$$

$$\Delta w(\tau_k) + \frac{p_k}{1-c} w(\tau_k - \sigma + h) \leq 0,\ k \in N$$

which is a contradiction. Indeed, the inequality (5) is of the form $(N_{2,\leq})$. But, by Theorem 1(b), the inequality (5) can not have eventually positive solutions.

The proof of the theorem is complete.

As consequences of the above theorem, we formulate the following propositions, the first of which is an analogous to Corollary 1.

**Corollary 2** *Assume that* $0 \leq q(\sigma - h) \leq \frac{1}{e}$ *and that*

$$\liminf_{t \to +\infty} \sum_{t-(\sigma-h) \leq \tau_k \leq t} p_k \geq 1 - c.$$

*Then the conclusion of Theorem 2 holds.*

**Corollary 3** *Assume that* $q(\sigma - h) \geq \frac{1}{e}$ *and that*

$$\liminf_{t \to +\infty} \sum_{t-(\sigma-h) \leq \tau_k \leq t} p_k \geq 1 - c - \frac{1}{e}.$$

*Then the conclusion of Theorem 2 holds.*

Next will be the result in the case of single deviation of Eq.$(E_1)$.

**Theorem 3** *Assume that* $\sigma = h$ *and that*

$$\liminf_{t \to +\infty} [qh + \sum_{t \leq \tau_k \leq t+h} p_k] \geq 1 + c.$$

*Then Eq.* $(E_1)$ *is oscillatory.*

**Proof.** Let, for the sake of contradiction, $y(t)$ be an eventually positive solution solution of the equation $(E_1)$. Then, in view of Lemma 2, the function $z(t) = y(t) - cy(t-h)$ and $w(t) = z(t) - cz(t-h)$ are eventually positive solutions to the equation $(E_1)$. That is, w(t) satisfies

$$[w(t) - cw(t-h)]' + qw(t-h) = 0 \ , t \neq \tau_k \tag{6}$$

$$\Delta[w(\tau_k) - cw(\tau_k - h)] + p_k w(\tau_k - h) = 0, \ k \in N.$$

Integrating Eq. (6) from $t$ to $t+h$, we obtain

$$w(t+h) - w(t) - c[w(t) - w(t-h)] + \sum_{t \leq \tau_k \leq t+h} p_k w(\tau_k - h) + q \int_t^{t+h} w(s-h)ds = 0,$$

and equivalently

$$w(t+h) - w(t) - c[w(t) - w(t-h)] + \sum_{t \leq \tau_k \leq t+h} p_k w(\tau_k - h) + qhw(t+h-h) \leq 0. \tag{7}$$

Since w(t) is a decreasing function of t, we see that $w(\tau_k - h) > w(t)$ for $t \leq \tau_k \leq t+h$ and so, from (7) we derive

$$w(t)(-1 - c + \sum_{t \leq \tau_k \leq t+h} p_k + qh) < 0,$$

which implies that

$$qh + \sum_{t \leq \tau_k \leq t+h} p_k < 1 + c.$$

The obtained contradiction proves the theorem.

As a consequence of the above theorem, we formulate the following proposition, that is an analogous to Corollary 1.

**Corollary 4** *Assume that* $\sigma = h$ *and* $0 \leq qh \leq \frac{1}{e}$ *and that*

$$\liminf_{t \to +\infty} \sum_{t \leq \tau_k \leq t+h} p_k \geq 1 + c.$$

*Then:*

*(i) the equation (1) is oscillatory;*

*(ii) the inequality (2) has no eventually positive solutions;*

*(iii) the inequality (3) has no eventually negative solutions.*

We conclude with an example, which illustrates Theorem 3 and its Corollary 4.

**Example 1** *The neutral impulsive differential equation with* $\tau_{k+1} - \tau_k = 1, \ k \in N$

$$[y(t) - \frac{1}{2}y(t-1)]' = 0, \ t \neq \tau_k$$

$$\Delta[y(\tau_k) - \frac{1}{2}y(\tau_k - 1)] + \frac{3}{2}y(\tau_k - 1) = 0 \ , k \in N,$$

*for every* $t > \tau_0 = 0$ *satisfies the assumptions of Corollary 4 of Theorem 3, i.e.*

$\liminf_{t \to +\infty} [qh + \sum_{t \leq \tau_k \leq t+h} p_k] \geq 1 + c$, where $\sigma = h = 1, q = 0, \ c = \frac{1}{2}, \ p_k = p = \frac{3}{2}$.

*Hence, it has only oscillatory solutions. It is obvious that these solutions will be in the form of piece-vice constant functions* $y(t) = A_k$, *for* $t \in (\tau_{k-1}, \tau_k], \ k \in N, \ t > \tau_0 = 0$ *with initial function*

$$\varphi(t) = A_0, \ t \in [\tau_0 - 1, \tau_0,], \ A_0 \in R$$

*where the "pulsatile" coefficients* $A_k$ *are determined by the difference scheme*

$$\Delta[y(\tau_k) - \frac{1}{2}y(\tau_k - 1)] + \frac{3}{2}y(\tau_k - 1) = 0 \ , k \in N,$$

$$i.e. \quad A_{k+1} = y(\tau_{k+1}) = (1+c)A_k - (p+c)A_{k-1},$$

*where*

$$A_{k-1} = y(\tau_{k-1}), \quad A_k = y(\tau_k), \quad A_{-1} = A_0.$$

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(1)Department of Mathematics
University of Ioannina
451 10 Ioannina, Greece

(2)Department of Mathematics
Technical University
8800 Sliven, Bulgaria